

The theta characteristic of a branched covering

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Abstract

We give a group-theoretic description of the parity of a pull-back of a theta characteristic under a branched covering. It involves lifting monodromy of the covering to the semidirect product of the symmetric and Clifford groups, known as the Sergeev group. As an application, we enumerate torus coverings with respect to their ramification and parity and, in particular, show that the corresponding all-degree generating functions are quasimodular forms.

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1. Introduction

1.1. Statement of the problem

1.1.1. Let

$$f: \mathcal{C} \rightarrow \mathcal{B} \tag{1.1}$$

be a degree d map between smooth algebraic curves. Let L be a theta characteristic of \mathcal{B} , that is, a divisor such that

$$2L = K_{\mathcal{B}},$$

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where $K_{\mathcal{B}}$ is the canonical class of \mathcal{B} . For example, if \mathcal{B} is a rational curve then any divisor of degree -1 is a theta characteristic. We will allow disconnected curves \mathcal{C} .

Throughout this paper we will assume that f has only *odd ramifications*, that is, its local form near any point of \mathcal{C} is

$$z \mapsto z^k, \quad k = 1, 3, 5, \dots$$

Under this assumption, one can define the pull-back of L under f as follows. Let ω be a meromorphic 1-form on \mathcal{B} such that $(\omega) = 2L$. All zeros of the 1-form $f^*\omega$ have even multiplicity and so we can set

$$f^*L = \frac{1}{2}(f^*\omega).$$

1.1.2. A theta characteristic L has a deformation invariant, namely its parity

$$p(L) = \dim \mathcal{O}(L) \pmod{2}. \quad (1.2)$$

This invariant has been studied from various points of view both classically and recently, see [2,7,15].

The main question addressed in this paper is how the parity of f^*L is related to the parity of L and the topological data of the branched covering f . The answer is given in Theorem 1. It involves the combinatorics of the Sergeev group $\mathcal{C}(d)$ defined in Section 1.3.2.

1.1.3. The particular case of a special importance to us will be the case when $\mathcal{B} = \mathbb{T}^2$ is a standard torus and $L = 0$. Using the representation theory of the Sergeev group, we enumerate the branched coverings

$$f: \mathcal{C} \rightarrow \mathbb{T}^2, \quad (1.3)$$

with given ramification with respect to the parity of their theta characteristic. This is done in Theorem 2.

As a corollary, we prove that the natural all-degree generating function for these numbers is a quasimodular form, refining a result of [3,5,13], see Corollary 2. This quasimodularity is a very useful property for the $d \rightarrow \infty$ asymptotic enumeration of coverings along the lines of [6]. It gives a way to compute the volumes of the corresponding strata in the moduli space of holomorphic differentials [14]. This application was the main motivation for the present work.

1.2. Johnson–Thurston formula

1.2.1. Every theta characteristic L determines a quadratic form q_L on the group

$$J_2(\mathcal{B}) \cong H^1(\mathcal{B}, \mathbb{Z}/2)$$

of elements of order 2 in the Jacobian of \mathcal{B} by the following rule:

$$q_L(\gamma) = p(L \otimes \gamma) + p(L). \quad (1.4)$$

The Arf invariant of this quadratic form is $p(L)$ itself, meaning that

$$(-1)^{p(L)} = 2^{-g(\mathcal{B})} \sum_{\gamma \in J_2(\mathcal{B})} (-1)^{q_L(\gamma)}, \quad (1.5)$$

assuming \mathcal{B} is connected. Here $g(\mathcal{B})$ is the genus of \mathcal{B} .

Throughout the paper, we identify $H^1(\mathcal{B})$ and $H_1(\mathcal{B})$ using the intersection pairing. By the Riemann–Mumford relation [1], the bilinear form associated with the quadratic form q_L is the intersection form.

1.2.2. We will use a different, purely topological, description of the form q_L which is due to Johnson [9] and Thurston [1]. Let ω be a meromorphic 1-form such that $(\omega) = 2L$. Away from the support of (ω) , we can introduce a local coordinate z such that

$$\omega = dz.$$

The transition functions between different charts will then be translations of \mathbb{C} , thus making \mathcal{B} a so-called *translation surface*.

Let Γ be an immersed closed curve on \mathcal{B} that avoids the support of (ω) . The translation surface structure determines the number of full rotations of the tangent vector to Γ . Taken modulo 2 this number is independent of the orientation of Γ . We will denote it by

$$\text{wind}(\Gamma) \in \mathbb{Z}/2.$$

The Johnson–Thurston formula can now be stated as follows:

$$\begin{aligned} q_L(\gamma) = & \text{wind}(\Gamma) + \# \text{ of components of } \Gamma \\ & + \# \text{ of self-intersections of } \Gamma, \end{aligned} \quad (1.6)$$

where Γ is a representative of the class γ . Note, in particular, that the right-hand side of (1.6) depends only on the homology class mod 2 of the curve Γ , which is elementary to check directly.

1.2.3. Given a map as in (1.1) and a class $\gamma \in H_1(\mathcal{C}, \mathbb{Z}/2)$, we set

$$\Delta_f(\gamma) = q_{f^*L}(\gamma) - q_L(f_*\gamma). \quad (1.7)$$

It is clear from (1.6) that this does not depend on the choice of the theta characteristic L and simply equals the number of “new” self-intersections of $f_*\gamma$, that is,

$$\begin{aligned} \Delta_f(\gamma) = & \# \text{ of self-intersections of } f(\Gamma) \\ & - \# \text{ of self-intersections of } \Gamma, \end{aligned} \quad (1.8)$$

where Γ is a representative of γ .

1.3. The Sergeev group

1.3.1. The quadratic form (1.8) is a purely combinatorial invariant of a branched covering (1.1) and a homology class $\gamma \in H_1(\mathbb{C}, \mathbb{Z}/2)$. Our first result is a group-theoretic interpretation of this invariant.

Every element $\gamma \in J_2(\mathbb{C})$ defines a unramified double cover $\mathcal{C}_\gamma \rightarrow \mathbb{C}$, and hence, by composing with f , a degree $2d$ cover

$$f_\gamma: \mathcal{C}_\gamma \rightarrow \mathcal{B}. \quad (1.9)$$

Let

$$\sigma: \mathcal{C}_\gamma \rightarrow \mathcal{C}_\gamma \quad (1.10)$$

be the involution permuting the sheets of $\mathcal{C}_\gamma \rightarrow \mathbb{C}$. The monodromy of the branched covering (1.9) lies in the centralizer of a fixed-point-free involution σ inside the symmetric group $S(2d)$.

In other words, the monodromy of f_γ defines a homomorphism

$$\mathbf{M}_{f_\gamma}: \pi_1(\mathcal{B}^\times) \rightarrow \mathbf{B}(d), \quad (1.11)$$

where

$$\mathcal{B}^\times = \mathcal{B} \setminus \{\text{branchpoints}\} \quad (1.12)$$

and the group

$$\mathbf{B}(d) = \mathbf{S}(d) \ltimes (\mathbb{Z}/2)^d \quad (1.13)$$

is the centralizer of a fixed-point-free involution in $S(2d)$. This group is also known as the Weyl group of the root systems B_d and C_d and has a convenient realization by signed permutations, that is, by automorphisms of $\{\pm 1, \dots, \pm d\}$ commuting with the involution $k \mapsto -k$.

1.3.2. The group $\mathbf{B}(d)$ has a remarkable central extension

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbf{C}(d) \rightarrow \mathbf{B}(d) \rightarrow 0 \quad (1.14)$$

which we will call the *Sergeev group* in honor of A. Sergeev who recognized in [17] its importance in the representation theory of superalgebras and its relation to the projective representations of the symmetric groups.

The Sergeev group $\mathbf{C}(d)$ is the semidirect product

$$\mathbf{C}(d) \cong \mathbf{S}(d) \ltimes \text{Cliff}(d), \quad (1.15)$$

where $\text{Cliff}(d)$ is the Clifford group generated by the involutions ξ_1, \dots, ξ_d and a central involution ϵ subject to the relation

$$\xi_i \xi_j = \epsilon \xi_j \xi_i, \quad i \neq j. \quad (1.16)$$

The group $S(d)$ acts on $\text{Cliff}(d)$ by permuting the ξ_i 's. Setting $\epsilon = 1$, gives the group $B(d)$.¹

1.3.3. The symmetric group $S(d)$ is naturally a subgroup of $B(d)$ and $C(d)$. We will call the corresponding elements of $B(d)$ and $C(d)$ pure permutations. Let

$$\delta \in \pi_1(\mathcal{B}^\times)$$

be a loop encircling one of the branchpoints. By construction, the monodromy $\mathbf{M}_{f_\gamma}(\delta)$ is conjugate in $B(d)$ to a pure permutation with odd cycles. We have the following well-known²

Lemma 1. *Let $g \in B(d)$ be conjugate to a pure permutation with odd cycles. Then only one of the two preimages of g in $C(d)$ is conjugate to a pure permutation.*

We will call this distinguished lift of $g \in B(d)$ to the Sergeev group the *canonical lift* of g . Given a homomorphism (1.11), we say that it has a canonical lift to the Sergeev group $C(d)$, if there exists a homomorphism

$$\widehat{\mathbf{M}}_{f_\gamma} : \pi_1(\mathcal{B}^\times) \rightarrow C(d), \quad (1.17)$$

that covers \mathbf{M}_{f_γ} , such that for all loops δ as above $\widehat{\mathbf{M}}_{f_\gamma}(\delta)$ is the canonical lift of $\mathbf{M}_{f_\gamma}(\delta)$. We define

$$\text{Lift}(f_\gamma) = \begin{cases} 0, & \mathbf{M}_{f_\gamma} \text{ has a canonical lift,} \\ 1, & \text{otherwise.} \end{cases} \quad (1.18)$$

1.3.4. In practice, this means the following. The group $\pi_1(\mathcal{B}^\times)$ is a group with one relation of the form

$$\prod_{i=1}^g [\alpha_i, \beta_i] \prod \delta_i = 1, \quad (1.19)$$

where the loops α_i and β_i follow the standard cycles of \mathcal{B} and the loops δ_i encircle the branchpoints of f . The square brackets in (1.19) denote the commutator

$$[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}.$$

Note that a commutator always has a canonical lift to any central extension.

Since we insist that all elements $\widehat{\mathbf{M}}_{f_\gamma}(\delta_i)$ are canonical lifts, this fixes the image of the left-hand side of (1.19) under the map $\widehat{\mathbf{M}}_{f_\gamma}$ completely. It follows that

$$\prod_{i=1}^g [\widehat{\mathbf{M}}_{f_\gamma}(\alpha_i), \widehat{\mathbf{M}}_{f_\gamma}(\beta_i)] \prod_i \widehat{\mathbf{M}}_{f_\gamma}(\delta_i) = \epsilon^{\text{Lift}(f_\gamma)}. \quad (1.20)$$

¹ Traditionally, the Sergeev group is defined slightly differently, namely, one sets $\xi_i^2 = \epsilon$ instead of $\xi_i^2 = 1$. While this leads to a nonisomorphic group, the representation theory is the same.

² The proof follows from observing that the commutator of a k -cycle $(123 \dots k)$ with $\xi_1 \xi_2 \dots \xi_k$ equals ϵ^{k-1} .

1.3.5. Our group-theoretic description of the invariant (1.7) is given in the following

Theorem 1. *We have*

$$\Delta_f(\gamma) = \text{Lift}(f_\gamma), \quad (1.21)$$

or, in other words,

$$(q_{f^*L}(\gamma) - q_L(f_*\gamma) = 0) \Leftrightarrow \left(\begin{array}{c} \text{monodromy of } f_\gamma \text{ has a} \\ \text{canonical lift to Sergeev group} \end{array} \right).$$

1.4. Torus coverings

1.4.1. We now specialize the case (1.3) of a branched covering of the torus. Let the ramification data of f be specified by a collection

$$M = (\mu^{(1)}, \mu^{(2)}, \dots) \quad (1.22)$$

of partitions of d into odd parts. We take $L = 0$ and set

$$p(f) = p(f^*0).$$

There is a classical method to enumerate coverings (1.3) in terms of characters of the symmetric group, see, for example, [10]. Our goal is to refine this enumeration taking into account the parity $p(f)$. This will involve characters of the Sergeev group. There exists a very close relation between these characters and characters of projective representations of the symmetric group, see [11,17–20].

1.4.2. Introduce the following $\mathbb{Z}/2$ -grading in the group $\mathbb{C}(d)$:

$$\deg \xi_i = 1, \quad \deg g = \deg \epsilon = 0, \quad g \in \mathbb{S}(d). \quad (1.23)$$

We will be interested in the irreducible $\mathbb{Z}/2$ -graded modules (also known as *supermodules*) of $\mathbb{C}(d)$ in which the central element ϵ acts nontrivially. Such modules are indexed by *strict* partitions λ of d . By definition, a partition λ is strict if

$$\lambda_1 > \lambda_2 > \dots \geq 0,$$

that is, if no nonzero part of λ is repeated.

Let V^λ denote the irreducible $\mathbb{C}(d)$ -supermodule corresponding to a strict partition λ . Let

$$g \in \mathbb{S}(d) \subset \mathbb{C}(d)$$

be a permutation with odd cycles. The conjugacy class of g acts in V^λ as multiplication by a constant, known as the *central character* of V^λ . It will be denoted by $\mathbf{f}_\mu(\lambda)$, where μ is the cycle type of g .³

³ To simplify the notation, we do not decorate $\mathbf{f}_\mu(\lambda)$ with any additional symbols to distinguish it from the identical notation used for the central characters of the symmetric group in e.g. [16]. Same remark applies to the functions \mathbf{p}_k

1.4.3. It is very convenient to omit parts equal to 1 from the odd partitions μ in (1.22) and in the label of $\mathbf{f}_\mu(\lambda)$. We also set

$$\mathbf{f}_\mu(\lambda) = 0, \quad |\mu| > |\lambda|.$$

With these conventions, it can be shown that $\mathbf{f}_\mu(\lambda)$ is in fact a *polynomial* in λ in the following precise sense. By definition, set

$$\mathbf{p}_k(\lambda) = \sum_i \lambda_i^k - \frac{\zeta(-k)}{2}, \quad k = 1, 3, 5, \dots, \quad (1.24)$$

where the constant is introduced for later convenience (compare with (1.28)). Denote by Λ the algebra

$$\Lambda = \mathbb{Q}[\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_5, \dots], \quad (1.25)$$

that the polynomials (1.24) generate. It is known as the *algebra of supersymmetric functions*.

It is known, see Section 6 in [8], that

$$\mathbf{f}_\mu \in \Lambda,$$

and moreover

$$\mathbf{f}_\mu = \frac{1}{z_\mu} \mathbf{p}_\mu + \dots \quad (1.26)$$

Here dots denote lower degree terms, $\mathbf{p}_\mu = \prod \mathbf{p}_{\mu_i}$, and

$$z_\mu = \prod_k k^{\rho_k} \rho_k!,$$

where $\mu = (3^{\rho_3}, 5^{\rho_5}, \dots)$. The formula (1.26) is the Sergeev group analog of a result of Kerov and Olshanski [12].

1.4.4. Our next result gives an enumeration of nonisomorphic branched covering (1.3) with given ramification M with respect to their parity $p(f)$. Each covering appears with a weight which is the reciprocal of the order of its automorphism group. Such weighting is standard when counting some objects up to isomorphism.

Theorem 2. *We have*

$$\sum_f \frac{(-1)^{p(f)} q^{\deg f}}{|\text{Aut}(f)|} = 2^{\chi(\mathbb{C})/2} \sum_{\text{strict } \lambda} (-1)^{\ell(\lambda)} q^{|\lambda|} \prod \mathbf{f}_{\mu(i)}(\lambda), \quad (1.27)$$

below and to algebra Λ they generate. Since the symmetric group analogs of these objects will not appear in this paper, this should not lead to confusion.

where the summation is over all nonisomorphic coverings f with specified ramification $\mu^{(i)}$ over fixed points $q_i \in \mathbb{T}^2$.

Here $\ell(\lambda)$ denotes the length of λ , that is, the number of its nonzero parts and $\chi(\mathcal{C})$ is the Euler characteristic of the cover, which can be computed from the ramification data (1.22) as follows:

$$\chi(\mathcal{C}) = \sum_k (\ell(\mu^{(k)}) - |\mu^{(k)}|).$$

1.4.5. In particular, in the unramified case $M = \emptyset$ we get the following series:

$$(q)_\infty = \prod_{n>0} (1 - q^n) = \sum_{\text{strict } \lambda} (-1)^{\ell(\lambda)} q^{|\lambda|}.$$

Set, by definition,

$$F_M(q) = \frac{1}{(q)_\infty} \sum_f \frac{(-1)^{p(f)} q^{\deg f}}{|\text{Aut}(f)|},$$

where the sum over f is the same as in Theorem 2. The series $F_M(q)$ enumerates coverings without unramified connected components.

As a corollary to Theorem 2, we obtain the following quasimodular property of the series $F_M(q)$. Let

$$E_k(q) = \frac{\zeta(-k+1)}{2} + \sum_n \left(\sum_{d|n} d^{k-1} \right) q^n, \quad k = 2, 4, 6, \dots, \quad (1.28)$$

denote the Eisenstein series of weight k . These series generate the algebra

$$QM_* = \mathbb{C}[E_2, E_4, E_6]$$

known as the algebra of *quasimodular forms*. This algebra is graded by the weight. We will denote by QM_k the graded component of weight k and by $QM_{\leq k}$ the filtered component of weight at most k .

Define the weight of a partition μ by

$$\text{wt } \mu = |\mu| + \ell(\mu),$$

where $\ell(\mu)$ is the length of μ , and set $\text{wt } M = \sum \text{wt } \mu^{(k)}$. We have the following

Corollary 2.

$$F_M(q) \in QM_{\leq \text{wt } M}. \quad (1.29)$$

The analogous generating series without the $(-1)^{p(f)}$ factor is known to be quasimodular form of the same weight by the result of [3]. It follows that the generating series for even and odd coverings separately are quasimodular.

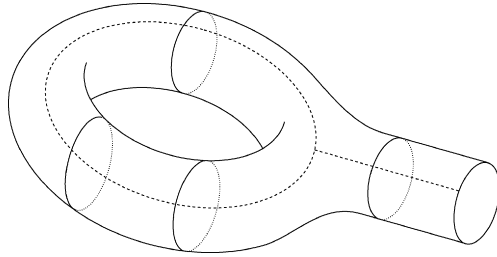


Fig. 1. A decomposition into tubes and pairs of pants.

2. Proof of Theorem 1

2.1. The proof of Theorem 1 will be based on the usual TQFT technique of cutting the punctured curve \mathcal{B}^\times into tubes and pairs of pants, see, for example, Fig. 1, in which a punctured torus is decomposed into 3 tubes and one pair of pants. The dashed line in Fig. 1 cuts each tube and each pair of pants into a simply-connected piece. This dashed line will be useful for keeping track of how the pieces of \mathcal{B}^\times fit together and for understanding the coverings of \mathcal{B}^\times .

By a choosing a suitable decomposition of \mathcal{B}^\times , we will achieve that the curve $f_*(\gamma)$ takes a particularly simple standard form on each piece. These standard pieces will have a simple interpretation in terms of the Sergeev group $\mathbf{C}(d)$.

2.2. Concretely, we require that on all pairs of pants the curve $f_*(\gamma)$ has the form shown in Fig. 2, while on the tubes it has one of the 4 forms shown in Fig. 3. Note that, with the exception of case (d) in Fig. 3, the curve $f_*(\gamma)$ stays in a small neighborhood of the dashed line. As a subcase, case (d) includes a circle around the tube with no vertical lines. Also note that the punctures of \mathcal{B}^\times should be capped off by tubes of the form (c) with no vertical lines remaining.

It is clear from this description that the number $\Delta_f(\gamma)$ of new self-intersections of $f_*(\gamma)$ equals the number of tubes of type (b) such that the two intersecting branches of $f_*(\gamma)$ come from different sheets of the covering.

2.3. We now translate these pictures into the language of branched coverings. Consider the monodromy around the neck ν of a tube or of a pair of pants, see Fig. 4. We choose the base

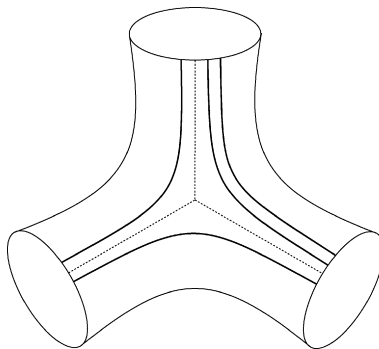


Fig. 2. The standard pair of pants.

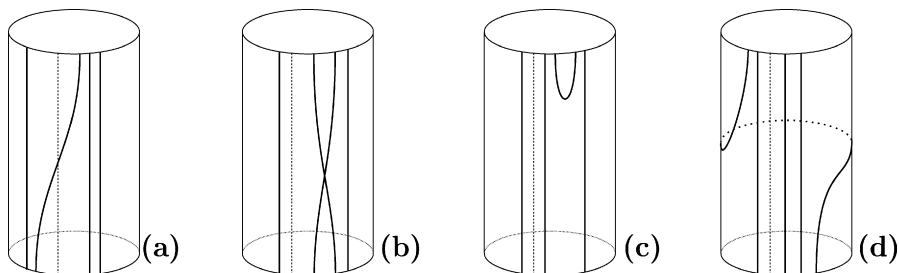
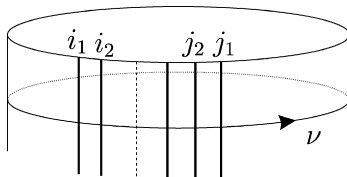


Fig. 3. The standard tubes.

Fig. 4. Monodromy around the neck ν .

point away from the dashed line and study how the different sheets of f_γ are glued together across the dashed line.

Let $g \in \mathbf{S}(d)$ be the monodromy of f around ν , that is, the permutation that describes how the sheets of f are identified across the dashed line. By construction, the monodromy $\mathbf{M}_{f_\gamma}(\nu) \in \mathbf{B}(d)$ is a signed version of g . More precisely, a sign appears whenever the lift of the monodromy path crosses the curve γ . Let us label the branches of $f_*(\gamma)$ by the sheets of f that they come from, as in Fig. 4. We choose

$$\widehat{\mathbf{M}}_{f_\gamma}(\nu) = \xi_{j_1} \xi_{j_2} \cdots g \cdots \xi_{i_2} \xi_{i_1} \quad (2.1)$$

as the lift of $\mathbf{M}_{f_\gamma}(\nu)$ to the Sergeev group.

For example, the pair of pants from Fig. 2 reflects the following multiplication rule:

$$(\xi_{j_1} \xi_{j_2} \cdots g \cdots \xi_{i_2} \xi_{i_1}) \cdot (\xi_{i_1} \xi_{i_2} \cdots h \cdots \xi_{k_2} \xi_{k_1}) = \xi_{j_1} \xi_{j_2} \cdots gh \cdots \xi_{k_2} \xi_{k_1}.$$

Similarly, the type (a) tube corresponds to the identity

$$g \xi_i = \xi_{g(i)} g,$$

while the type (c) tube is a geometric version of $\xi_i^2 = 1$.

2.4. Two necks can be glued together when the two monodromies are conjugate. The conjugating element gives the gluing rule. This conjugating element can be viewed as a new kind of tube. If it is a pure permutation $h \in \mathbf{S}(d)$, then this tube looks rather dull, see Fig. 5, because of the relation

$$h \cdot (\xi_{j_1} \xi_{j_2} \cdots g \cdots \xi_{i_2} \xi_{i_1}) \cdot h^{-1} = \xi_{h(j_1)} \xi_{h(j_2)} \cdots gh^{-1} \cdots \xi_{h(i_2)} \xi_{h(i_1)}.$$

If, however, this conjugating element is ξ_i for some i then the corresponding tube is of type (d).

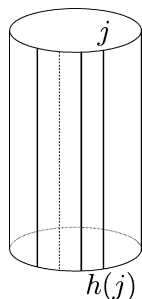


Fig. 5. The glue tube.

2.5. Finally, the tube of type (b) is a plain violation of the Sergeev group law, unless the two intersecting branches come from the same sheet of the covering. Each instance of such violation contributes a factor of ϵ to the product (1.20). Hence, in total, the product (1.20) equals $\epsilon^{\Delta_f(\gamma)}$. This completes the proof of Theorem 1.

3. Torus coverings

3.1. Proof of Theorem 2

3.1.1. We will begin with a quick review of the relation between branched coverings and characters of groups, see, for example, [4,10].

Let G be a finite group and let a conjugacy class $C_i \subset G$ be specified for each element of a finite set of points $\{r_i\} \subset \mathbb{T}^2$. A natural question associated to these data is to count the homomorphisms

$$\psi : \pi_1(\mathbb{T}^2 \setminus \{r_i\}) \rightarrow G$$

sending the conjugacy class of a small loop around $\{r_i\}$ into the conjugacy class C_i . By definition, set

$$h_G(C_1, C_2, \dots) = \frac{\# \text{ of such } \psi}{|G|}. \quad (3.1)$$

In the numerator here, one counts all homomorphisms ψ without any equivalence relation imposed. Alternatively, the number h_G can be interpreted as the automorphism weighted count of homomorphisms up the action of G by conjugation.

A classical formula, which goes back to Burnside, for the number h_G is the following:

$$h_G(C_1, C_2, \dots) = \sum_{\lambda \in \hat{G}} \prod \mathbf{f}_{C_i}(\lambda), \quad (3.2)$$

where \hat{G} is the set of irreducible complex representations of G and $\mathbf{f}_C(\lambda)$ is the central character of the representation λ .

3.1.2. The cases of interest to us will be

$$G = S(d), B(d), B_0(d), C(d), C_0(d), \quad (3.3)$$

where

$$C_0(d) \subset C(d), \quad B_0(d) \subset B(d)$$

are subgroups of index two formed by even elements with respect to the grading (1.23).

The conjugacy classes of interests to us all come from $S(d)$ and are indexed by partitions of μ of d into odd parts. We will denote by

$$h_G(M) = h_G(C_{\mu^{(1)}}, C_{\mu^{(2)}}, \dots)$$

the count of homomorphisms corresponding to the ramification data (1.22).

3.1.3. All groups (3.3) have a natural map to $S(d)$ and we will denote by

$$h_G(M; \phi)$$

the count of those ψ that composed with $G \rightarrow S(d)$ yield some fixed homomorphism

$$\phi : \pi_1(\mathbb{T}^2 \setminus \{r_i\}) \rightarrow S(d). \quad (3.4)$$

Lemma 3. For any ϕ , we have

$$h_{B(d)}(M; \phi) = \frac{1}{d!} \prod_i 2^{2g(\mathcal{C}_i)-1}, \quad (3.5)$$

where the product ranges over all connected components \mathcal{C}_i of the curve \mathcal{C} and $g(\mathcal{C}_i)$ denotes the genus of \mathcal{C}_i .

Proof. If \mathcal{C} is connected, then the set $J_2(\mathcal{C})$ is in a natural bijection with the orbits of the action of $(\mathbb{Z}/2)^d \subset B(d)$ by conjugation on all lifts of ϕ to $B(d)$. Hence, there are $2^{2g(\mathcal{C})}$ such orbits.

The stabilizer of the $(\mathbb{Z}/2)^d$ -action is isomorphic to $\mathbb{Z}/2$ for any orbit. Indeed, any element in the stabilizer commutes with the subgroup of $S(d)$ generated by ϕ . Since this group is transitive, it follows that the stabilizer is the group $\mathbb{Z}/2$ generated by the involution $k \mapsto -k$. Hence the number of elements in each orbit is 2^{d-1} .

Multiplying 2^{d-1} by $2^{2g(\mathcal{C})}$ and dividing by the order of the group gives (3.5). The generalization to a disconnected curve \mathcal{C} is straightforward. \square

3.1.4. Our goal now is to specialize Theorem 1 to the case $\mathcal{B} = \mathbb{T}^2$ and $L = 0$. For $L = 0$, the quadratic form $q_L(v)$, $v \in H_1(\mathbb{T}^2, \mathbb{Z}/2)$, takes a very simple form, namely

$$q_0(v) = \begin{cases} 0, & v = 0, \\ 1, & v \neq 0. \end{cases} \quad (3.6)$$

We now make the following simple observation:

Lemma 4. For $\gamma \in H_1(\mathbb{C}, \mathbb{Z}/2)$, the push-forward $f_*\gamma \in H_1(\mathbb{T}^2, \mathbb{Z}/2)$ vanishes if and only if the monodromy of f_γ lies in the group $B_0(d)$.

We now see that Theorem 1 specializes to the group-theoretic description of the quadratic form q_{f^*L} summarized in the following table:

Monodromy of f_γ lifts to		$q_{f^*L}(\gamma)$
$B_0(d)$	$C(d)$	
yes	yes	0
yes	no	1
no	yes	1
no	no	0

3.1.5. This table translates into the following:

Proposition 5. Let ϕ be a homomorphism (3.4) corresponding to a covering f as in (1.3). Then

$$\frac{(-1)^{p(f)}}{d!} = 2^{\chi(\mathbb{C})/2} (h_{B(d)}(M; \phi) - h_{B_0(d)}(M; \phi) - h_{C(d)}(M; \phi) + h_{C_0(d)}(M; \phi)). \quad (3.7)$$

Proof. Let ψ be a lift of ϕ to $B(d)$ corresponding to some class $\gamma \in H_1(\mathbb{C}, \mathbb{Z}/2)$. If the image of ψ lies in $B_0(d)$, then it contributes twice as much to $h_{B_0(d)}(M; \phi)$ because the order of the group is smaller by a factor of 2.

Suppose that ψ has a canonical lift to $C(d)$. Then, in fact, there are 4 such lifts because the monodromy around the two periods of the torus can be lifted arbitrarily. Since $|C(d)|/|B(d)| = 2$, the homomorphism ψ contributes twice as much to $h_{C(d)}(M; \phi)$ as it does to $h_{B(d)}(M; \phi)$.

Finally, if ψ lifts to $C_0(d)$, its contribution to $h_{C_0(d)}(M; \phi)$ is 4 times as much by the combination of two arguments.

Using the above table, one checks that in all cases these contribution of ψ to the alternating expression in the right-hand side of (3.7) combine to $(-1)^{q_{f^*L}(\gamma)}$. Now it remains to apply Lemma 3 and formula (1.5). \square

3.1.6. It is clear from (3.2) that

$$h_{C(d)}(M) - h_{B(d)}(M) = \sum_{\lambda \in C(d)_-^\wedge} \prod \mathbf{f}_{\mu^{(i)}}(\lambda), \quad (3.8)$$

where \mathbf{f}_μ is the central character corresponding to an odd partition μ of d and $C(d)_-^\wedge$ denotes the set of irreducible representations of $C(d)$ in which the central element ϵ acts as multiplication by -1 .

It follows from (3.7) and (3.8) that

$$2^{-\chi(\mathbb{C})/2} \sum_{\deg f=d} \frac{(-1)^{p(f)}}{|\text{Aut } f|} = \left(\sum_{\lambda \in C_0(d)_-^\wedge} \prod \mathbf{f}_{\mu^{(i)}}(\lambda) \right) - \left(\sum_{\lambda \in C(d)_-^\wedge} \prod \mathbf{f}_{\mu^{(i)}}(\lambda) \right), \quad (3.9)$$

where the summation on the left is over nonisomorphic degree coverings f with ramification data M .

3.1.7. Consider the algebra

$$A(d) = \mathbb{C}[C(d)]/(\epsilon + 1).$$

The grading (1.23) makes it a semisimple associative superalgebra. The structure of this algebra, or equivalently, representations of $C(d)$ in which ϵ acts nontrivially, were first studied by Sergeev in [17] and in many papers since, see References. In particular, the irreducible supermodules V^λ of $A(d)$ are indexed by strict partitions λ of d .

3.1.8. If $\ell(\lambda)$ is even, then the corresponding summand of $A(d)$ is isomorphic, for some n , to the associative superalgebra

$$\mathfrak{gl}(n, n) = \left\{ \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \right\}, \quad \deg A_{ij} = i + j, \quad (3.10)$$

of all endomorphisms of the $\mathbb{Z}/2$ -graded vector space V^λ . In particular, V^λ is an irreducible representation of $C(d)$, which splits into two nonequivalent irreducible modules

$$V^\lambda = V_0^\lambda \oplus V_1^\lambda$$

as a module over the even subalgebra

$$A \supset A_0 = \mathbb{C}[C_0(d)]/(\epsilon + 1).$$

However, these two representations, being conjugate under the action of $C(d)$, have the same central character when restricted to permutations with odd cycles. Indeed, any permutation with odd cycles commutes with some odd element of $C(d)$, such as, for example, (123) commutes with $\xi_1\xi_2\xi_3$.

It follows that when $\ell(\lambda)$ is even the contribution of V^λ to the first sum in (3.9) is twice its contribution to the second sum in (3.9).

3.1.9. If $\ell(\lambda)$ is odd then the corresponding summand of $A(d)$ is isomorphic, for some n , to the associative superalgebra

$$\mathfrak{q}(n) = \left\{ \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix} \right\}, \quad \deg A_i = i. \quad (3.11)$$

It follows that the corresponding supermodule V^λ is a direct sum of two nonequivalent representations of $C(d)$, which however, become equivalent after restriction to $C_0(d)$. Therefore, the contribution of such λ to the second sum in (3.9) is precisely twice its contribution to the first sum in (3.9).

3.1.10. We can summarize this discussion as follows:

$$\sum_{\deg f=d} \frac{(-1)^{p(f)}}{|\text{Aut } f|} = 2^{x(\mathbb{C})/2} \sum_{\text{strict partitions } \lambda \text{ of } d} (-1)^{\ell(\lambda)} \prod \mathbf{f}_{\mu(i)}(\lambda), \quad (3.12)$$

which is equivalent to the statement of Theorem 2.

3.2. Quasimodularity of $F_M(q)$

3.2.1. For $f \in \Lambda$, let

$$\langle f \rangle = \frac{1}{(q)_\infty} \sum_{\text{strict } \lambda} (-1)^{\ell(\lambda)} q^{|\lambda|} f(\lambda)$$

denote the average of f with the respect to the weight $(-1)^{\ell(\lambda)} q^{|\lambda|}$ on strict partitions. To prove Corollary 2, it clearly suffices to show that

$$\langle \mathbf{p}_\mu \rangle \in \mathcal{QM}_{\text{wt } \mu}. \quad (3.13)$$

The proof of (3.13) is straightforward and follows [3].

3.2.2. Let t_1, t_3, t_5, \dots be formal variables. We have

$$\begin{aligned} & \frac{1}{(q)_\infty} \sum_{\text{strict } \lambda} (-1)^{\ell(\lambda)} q^{|\lambda|} \exp\left(\sum t_k \mathbf{p}_k(\lambda)\right) \\ &= \exp\left(-\frac{1}{2} \sum t_k \zeta(-k)\right) \prod_{n>0} \frac{(1 - q^n \exp(\sum t_k n^k))}{(1 - q^n)}. \end{aligned} \quad (3.14)$$

This is a generating function for the averages (3.13). The logarithm of (3.14) equals

$$-\frac{1}{2} \sum t_k \zeta(-k) - \sum_{m,n} \frac{q^{mn}}{m} \left[\exp\left(m \sum t_k n^k\right) - 1 \right]. \quad (3.15)$$

For an odd partition μ , the coefficient of $\prod t_{\mu_i}$, in the expansion of (3.15) equals, up to a constant factor,

$$\delta_{\ell(\mu),1} \frac{\zeta(-\mu_1)}{2} + \sum_{n,m} q^{mn} m^{\ell(\mu)-1} n^{|\mu|} = \left(q \frac{d}{dq}\right)^{\ell(\mu)-1} E_{|\mu|-\ell(\mu)+2} \in \mathcal{QM}_{\text{wt } \mu}.$$

This is because the operator $q \frac{d}{dq}$ preserves the algebra \mathcal{QM}_* and increases weight by 2, which is proved in [13] and can be also seen, for example, from the heat equation for the genus 1 theta-functions.

Corollary 2 now follows by exponentiation.

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